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A method of producing a shear flow in a stratified fluid

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When the end of a long closed horizontal tube containing a stratified fluid is raised, a laminar accelerating flow begins. The flow is two-dimensional in the central portion of the tube, and, in this region, it is predictable, at least until the onset of instability. The occurrence of instability, its nature, and the subsequent transition to turbulence, are described qualitatively. The apparatus may be used for the study of a variety of other internal hydraulic phenomena with applications to meteorology and oceanography.

1. Introduction

In 1883 Osborne Reynolds read a paper to the Royal Society entitled 'An experimental investigation of the circumstances which determine whether the motion of water shall be direct or sinuous, and of the law of resistance in parallel channels' and described briefly experiments 'with two streams in opposite directions in the same tube. A glass tube, 5 ft. long and 1·2 in. in diameter, having its ends slightly bent up, was half filled with bisulphide of carbon, and then filled up with water and both ends corked. The bisulphide was chosen as being a limpid liquid but little heavier than water and completely insoluble, the surface between the two liquids being clearly distinguishable. When the tube was placed in a horizontal direction, the weight of the bisulphide caused it to spread along the lower half of the tube, and the surface of separation of the two liquids extended along the axis of the tube. On one end of the tube being slightly raised the water would flow to the upper end and the bisulphide fall to the lower, causing opposite currents along the upper and lower halves of the tube, while in the middle of the tube the level of the surface of separation remained unaltered.

'The particular purpose of this investigation was to ascertain whether there was a critical velocity at which waves or sinuosities would show themselves in the surface of separation.

'It proved a very pretty experiment and completely answered its purpose.'†

The purpose of this paper is to describe qualitatively some experiments which have been made in an apparatus similar to that used by Reynolds, and using a similar technique to initiate the flow. The apparatus lends itself to a wide variety of experiments if instead of confining attention to a two-fluid system, fluids of continuous, but gravitationally stable, stratifications are used. Perhaps the most

† A similar study was made by Mittendorf (1961), and this is mentioned in the paper by Macagno & Rouse (1961), who call it the Helmholtz experiment.

important application of the technique is to a study of the onset of instability in a heterogeneous shear flow. Several of the velocity and density distributions of which the stability has been successfully treated theoretically, may be produced in the apparatus. Another application is to the study of stratified shear flow over an obstacle (see the appendix by Bretherton, Hazel, Thorpe & Woods to the paper by Hazel, 1967) and the accompanying critical layer absorption, lee waves and blocking effects.

Before discussing the theory of the flow of a stratified fluid in a long tube tilted at an angle to the horizontal, a few remarks must be made about the observed flow. When the tube is suddenly tilted no Taylor instability has been seen, nor does the sudden rotation of the liquid about a horizontal axis produce any observable initial motion or instability. The flow, starting from rest, is parallel to the tube walls in the centre of the tube and measurements show that, except where influenced by viscosity, the flow accelerates uniformly, although the arrival of the flow at an end of the tube results in a surge which travels towards the centre of the tube, eventually destroying the laminar parallel flow if it has not itself already become unstable. The theory and the majority of observations are concerned with the flow near the centre of the tube.

2. Theory

We consider the motion of a stratified fluid from a state of rest between parallel planes, $z = \pm H/2$, inclined at an angle to the horizontal. The effect of lateral walls will be ignored and the x-axis taken parallel to the planes up the line of greatest slope, and the z-axis upwards (figure 1). The density of the fluid, ρ, is supposed to depend on co-ordinate z alone, and in accordance with the observations, the velocity $(u, 0, 0)$ is parallel to the boundaries. By continuity u is a function of z and time t alone. For simplicity we shall suppose that the fluid is inviscid (the solution including effects of viscosity is presented in the appendix). The equations of motion are

$$\rho \frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} - g\rho \sin \alpha \tag{1}$$

and

$$0 = -\frac{\partial p}{\partial z} - g\rho \cos \alpha, \tag{2}$$

where p is pressure and g the acceleration due to gravity.

Now since ρ is a function of z only, from (2) $\partial p/\partial z$ is a function only of z, and p is therefore equal to a function of z plus another function of x and t, and $\partial p/\partial x$ is a function of x and t only. But since u and ρ are functions of z and t only, from (1) $\partial p/\partial x$ is a function of t only. Integrating (1) we have

$$u = -\frac{1}{\rho}\int_0^t \frac{\partial p}{\partial x}\, dt - gt \sin \alpha, \tag{3}$$

using the condition that $u = 0$ at $t = 0$.

Since the ends of the tube are closed there is no net flux across any plane $x =$ constant, and hence

$$\int_{-H/2}^{H/2} u\, dz = 0.$$

Integrating (3) we find that

$$\int_0^t \frac{\partial p}{\partial x}\,dt = -gHt\sin\alpha \Big/ \int_{-H/2}^{H/2} \frac{dz}{\rho} \tag{4}$$

and so

$$u(z,t) = gt\sin\alpha\left(\frac{H}{\rho(z)\displaystyle\int_{-H/2}^{H/2}\frac{dz}{\rho}} - 1\right). \tag{5}$$

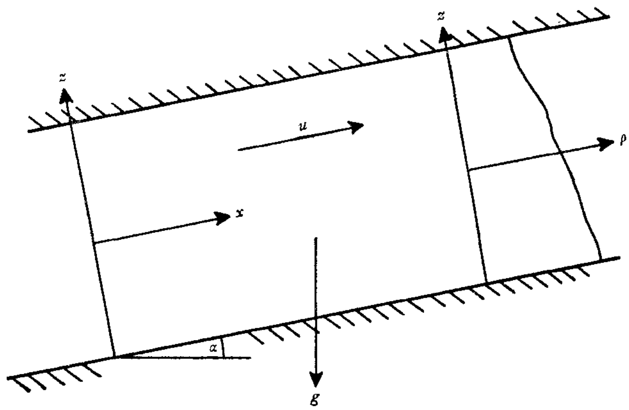

FIGURE 1. Notation.

If after a time τ the tube is suddenly restored to the horizontal position without disturbing the flow, the flow will continue to move with speed $u(z,\tau)$, there being no horizontal forces on the fluid, and the local Richardson number,

$$Ri = -\frac{g}{\rho}\frac{d\rho}{dz}\Big/\left(\frac{du}{dz}\right)^2,$$

is equal to

$$\frac{\rho^2}{H^2}\left[\int_{-H/2}^{H/2}\frac{dz}{\rho}\right]^2 \Big/ N^2\tau^2\sin^2\alpha, \tag{6}$$

where

$$N(z) = \left(-\frac{g}{\rho}\frac{d\rho}{dz}\right)^{\frac{1}{2}}$$

is the stability or Brunt-Väisälä frequency.

For fluids in which $\rho = \rho_0(1-\Delta f(z))$, where ρ_0 and Δ are constants, $\Delta \ll 1$, and $f(z) = -f(-z)$ is of order unity, u is approximately equal to $g\Delta f(z)t\sin\alpha$ and

$$Ri = \frac{1}{N^2\tau^2\sin^2\alpha}$$

where $N^2 = -g\Delta f'$. With the function $f(z) = az$, we reproduce a situation similar to that considered theoretically by Taylor (1931), Eliassen, Høiland & Riis (1953), and Case (1960) and which is predicted to be stable in the presence of one boundary at all positive Richardson numbers. If $f(z) = \tanh az$ we have approximately Holmboe's case (see Drazin & Howard, 1966, p. 77, for this and other solutions) and if

$$f(z) = \begin{cases} 1 & (H/2 \geqslant z \geqslant h), \\ z/h & (h \geqslant z \geqslant -h), \\ -1 & (-h \geqslant z \geqslant -H/2), \end{cases}$$

we have the case first considered by Taylor (1931) and Goldstein (1931) and more recently by Miles & Howard (1964). The steady flow in both these cases is known to be unstable to certain infinitesimal disturbances if $Ri < \frac{1}{4}$ provided that $H \to \infty$, and the critical wave-numbers are known in terms of the thickness of the interface. Howard (1964) has shown that the presence of boundaries exerts a stabilizing effect in the case of a homogeneous fluid with 'tanh' profile. Provided that the critical wave-number is sufficiently greater than $1/H$, the stability of the fluid will be only slightly affected by the presence of the boundaries.

The solution for two layers of fluid which fill the tube, the upper of depth h_1, density ρ_1, and speed u_1 and the lower similarity defined with suffices 2, $\rho_2 > \rho_1$, is easily found and is

$$\left.\begin{aligned} u_1 &= \frac{(\rho_2-\rho_1)\,gh_2 t\sin\alpha}{\rho_1 h_2+\rho_2 h_1}, \\ u_2 &= -\frac{(\rho_2-\rho_1)\,gh_1 t\sin\alpha}{\rho_1 h_2+\rho_2 h_1}. \end{aligned}\right\} \tag{7}$$

3. The apparatus and method of filling

A variety of experiments have been made in a tube of rectangular 10 cm × 3 cm section and 183 cm length made of perspex so that observation of the flow from all directions is possible (figure 2). The tube, A, is supported on a dexion frame

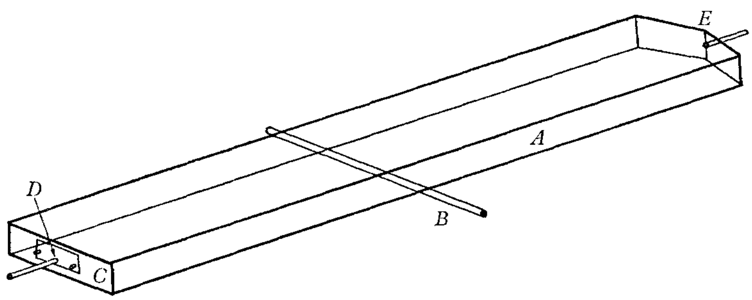

FIGURE 2. The apparatus.

and pivoted about a horizontal axis, B, a few centimetres below the centre of the tube. Two positions of mounting are possible with either the 3 cm or the 10 cm side vertical. One end of the tube, C, is closed by a perspex plate which is removable to facilitate cleaning and the mounting of obstacles within the tube. The plate has a $\frac{1}{4}$ in. diameter tube attached to it, and it is through this tube that the apparatus is filled, the inflow being diverted by a small plate of perspex, D, mounted just above the end plate, to avoid mixing. The other end of the apparatus, E, is closed and shaped to allow the easy removal of bubbles from the apparatus during filling. Another $\frac{1}{4}$ in. tube leads from this end to waste, and is closed by a clip when the apparatus has been filled.

It is found convenient to fill the tube from the bottom when its longest side is vertical. Two reservoirs mounted above the top of the tube supply liquid (usually brine solutions coloured by the addition of a few potassium permanganate crystals are used); the liquid passes into the apparatus under gravity. The method described by Oster (1965, p. 74, 'Instant' density gradient) has been found convenient for producing a constant density gradient. When filled, the tube is slowly rotated about the horizontal axis from its vertical position until it is horizontal. With care this may be done with very little mixing in the fluid and any density gradients considerably sharpened in the process. As the tube is so tilted, the dense fluid spreads along the bottom with a front like an internal surge (or nose in the terminology of Ellison & Turner, 1959). A similar, but inverted, surge is found at the front of the less dense fluid. The fluid takes about 4 min to reach a state of rest after the tube arrives at a horizontal position, and experiments made by tilting the tube may then commence. If two layers of fluids separated by a sharp interface are used and the tube tilted rapidly into this horizontal position, three-dimensional progressive waves are generated behind the advancing surges, breaking either at their crests or at their troughs, eventually one train travelling through the other in the direction of the fluid into which it is breaking. The waves do not appear to be long-crested and those observed at the interface between two fluids (water and brine) with density difference of $1{\cdot}44 \times 10^{-1}$ gm cm^{-3} had a wavelength of about 3·1 cm. They appear to be similar to the waves described by Townsend (1957, p. 372) in his discussion of Turner's experiments. These waves are most clearly seen when a thin layer of dye of intermediate density is placed at the interface between two clear liquids of different densities, and then both trains of waves can clearly be seen simultaneously, that moving with the upper fluid having sharp crests from which fluid is continuously being detached and fed into the upper fluid, and flat troughs, and the wave-train moving with the lower fluid having sharp troughs and flat crests. Figure 3*a*, plate 1, shows one train of these waves behind a surge at the front of the dense fluid which moves to the right, and figure 3*b*, plate 1, shows a later stage in which both wave-trains have arrived and move through one another. The front of the rapidly moving surge, when viewed from above, is found to have an irregular outline (figure 4, plate 2) rather similar in form to the profiles of a fluid front in Taylor instability photographed by Lewis (1950).

The dimensions of the apparatus were chosen to allow experiments in which the Richardson number in a fluid with a constant density gradient produced by brine could be reduced well below the value of $\frac{1}{4}$ in the hope of establishing the stability or instability, of the flow at such low Richardson numbers. There are several factors which must be taken into account in choosing the size of the apparatus to satisfy these requirements. The time, t_0, during which the flow in the centre of the tube is unaffected by the end walls is determined by the speed of the surges penetrating from the ends. This speed is not readily predictable, but an upper bound was taken to be given by the sum of the maximum fluid velocity and the speed of a solitary wave in a stationary fluid. The observed speeds were found to be less than this. During the time t_0 the fluid must achieve sufficiently low Richardson numbers as required, and moreover the effects of viscosity must be confined to a layer of thickness small in comparison with the tube height, H. The maximum density difference between the top and bottom of the fluid which may be obtained at normal room temperatures using brine is about 0·2 gm cm^{-3}.

A further factor is that instability, if it occurs, must have time to grow to an observable size during the time t_0. In many experiments the growth rate is not predictable in advance, at least when the disturbance achieves finite amplitude, and only previous experience can indicate what times are necessary. It is also clear that any statement about observed stability of a flow at a given Richardson number should also contain information about the limitations on the observable growth rate.

The present apparatus is not of sufficient length to allow the study of a class of flows which will be described later (§4.2), and a longer tube is now being constructed. Observations (§4.1) indicate that great care must be taken to avoid irregularities in the tube boundaries if precise experiments are to be made, but for instructional purposes an 'off the shelf' perspex or glass tube 4 or 5 ft. in length and 2 in. in diameter corked at both ends is quite adequate. For many demonstrations two immiscible fluids (for example water and paraffin or carbon tetrachloride or a mixture of the latter two) are suitable and the experiment may be repeated again and again without the time-consuming task of refilling the apparatus after the miscible fluids have mixed.

4. Experiments

A number of experiments have been made in the apparatus. Except for those concerned with the flow over an obstacle referred to earlier, all so far have been made in the tube tilted at an angle to the horizontal, the procedure being to tilt the tube rapidly by raising one end by hand through a prearranged distance until the top of the tube comes into contact with a rubber-protected piece of dexion (this takes about a quarter of a second) and to observe the flow in the tube and record events on 16 mm cine film running at about 64 frames per second through a Bolex camera. In some cases when both the wavelengths and growth rates of unstable disturbances are to be measured, a 35 mm Nikon camera with motor drive taking about 4 frames per second is used for 'long shots' from which wavelengths may be measured, whilst the Bolex is used to obtain photographs in 'close-up' of one or two waves from which both the growth of the waves and the time at which instability occurs after tilting the tube may be determined. These experiments are being continued but some of the results will be described qualitatively to illustrate the possibilities of the technique.

4.1. *Two miscible fluids with a sharp interface*

Observations of the flow in a two-layer system were made using a technique described by Baker (1966) in the tube with $H = 10$ cm. Two 0·002 in. diameter platinum wires were stretched vertically between the top and bottom of the tube, 0·5 cm apart, and in a plane normal to the tube axis. One wire was mounted halfway between the front and back walls of the tube. The two fluids, of equal depth, were distilled water and a mixture of water and methylated spirits, to both of which was added Thymol Blue, a pH indicator. Titration to the end-point was performed as described by Baker. A thin cylinder of brownish coloured fluid was produced around the central wire by impressing 6 V d.c. for a second or so across the wires with the filled tube in the horizontal position, the mixture of water and methylated spirits floating uppermost. The motion of the cylinder of dyed fluid was recorded on cine film when the tube was tilted through a small angle, and measurements of its position were compared with theoretical estimates derived from (7). These were found to be in very good agreement for the first 10 sec of motion, but after this the fluid accelerations were reduced, probably as the result of the viscous boundary layers at the lateral walls which, by this time, would have filled the tube.

The remainder of these experiments were made with brine, coloured by potassium permanganate, and fresh water. With fluids of equal depth and density difference, $\Delta\rho$, between 4×10^{-3} and $2\times 10^{-1}\,\mathrm{gm\,cm^{-3}}$ and an angle of tilt, α, between 5° and 30°, a few seconds after tilting the tube a beautiful array of waves suddenly appeared on the interface and rolled up in about a second (figure 5, plates 3–5). The waves appear to be stationary and two-dimensional with their crests normal to the direction of flow. They are found to grow spontaneously and almost simultaneously along the whole length of the tube, although waves in one part of the tube may grow slightly earlier than those at another part. For given values of $\Delta\rho$, α and H, the wavelength of the instability and the time, t_1, at which it first occurs, are remarkably repeatable. The most striking features of the observations are the regularity of the waves and the beautiful spiral form assumed by the disturbance before further break-up of the motion. At a density difference $\Delta\rho = 6{\cdot}02\times 10^{-2}\,\mathrm{gm\,cm^{-3}}$ and $\alpha = 8{\cdot}20°$, the instability is first observed 3·06 sec after the tube with $H = 10\,\mathrm{cm}$ is first tilted (the time, t_1, is measured as the sum of half the time taken to tilt the tube fully and the time measured from the instant the tube reaches its tilted position until that at which instability could first be detected) and has a wavelength of 3·9 cm. The fluid speed at this time is about $13\,\mathrm{cm/sec^{-1}}$.† The waves reach a maximum amplitude after a further 0·8 sec when the wave slope ($2\pi a/\lambda$ where $2a$ is the crest to trough height and λ the wavelength) is approximately 1·1.

The waves at first appear to be fairly sinusoidal in form, but distort as they grow into a spiral form whilst remaining fairly symmetrical about the original position of the interface. This distortion is accompanied by the accumulation of fluid of intermediate density on the side of the wave which ultimately rolls up, and considerable sharpening of the remainder of the interface. This accumulation is most strikingly demonstrated if a thin layer of coloured fluid of intermediate density is observed between two clear fluids, and in this case almost all the dyed fluid is moved into the regions where the roll-up occurs. The spiral form is seen to become unstable (presumably due to its being gravitationally unstable) by fluid breaking from one spiral band into another. The spiral itself was best produced with $H = 3\,\mathrm{cm}$ and it was never as clearly defined with $H = 10\,\mathrm{cm}$, but

† The Reynolds number, *Re*, at the onset of instability, based on the momentum thickness of the velocity transition region and the maximum velocity shear, is approximately

$$0{\cdot}28\left(\frac{\rho_2-\rho_1}{\rho_1+\rho_2}\right)\sin\alpha\left(\frac{g^2t^3}{\nu}\right)^{\frac{1}{2}}$$

and this is equal to 254 in the case discussed. This is well within the range examined by Freymuth (1966) for a homogeneous fluid, in which the transition to turbulence is found to be independant of *Re*.

this was possibly due to the sharper interfaces produced as a result of the larger area of the interface when $H = 3$ cm. With $H = 3$ cm, the spirals grew until they almost filled the whole depth of the tube, but with $H = 10$ cm the instability of the spirals resulted in a row of mixed cylinders or pencils of fluid as yet remote from the boundaries at $z = \pm H/2$. This row, and occasionally the similar row of rotating elements with $H = 3$ cm, was observed to become unstable under a mutual interaction of individual elements in which the pencils of fluid began to rotate around one another in pairs and, as they did so, to amalgamate (partly as the result of gravitational instability). The interface between pairs engaged in this 'secondary roll-up' was observed to develop waves and spiral rolls but of a much smaller scale than that first observed on the parallel flow,† the whole flow now assuming a pattern of large rolls with little rolls riding upon them. In the experiments the size of the 'secondary rolls' were such as to fill the tube completely and the whole flow was involved in their break-up. The 'secondary roll-up' in pairs may be related to the instability of a row of line vortices which, at least in an infinite fluid, has a maximum growth rate when the vortices roll up in pairs (Lamb 1932, §156), but there is no evidence that the pencils of fluid themselves are truly concentrations of vorticity, although at first sight they appear to be. When $H = 3$ cm the spirals block and arrest the flow in the upper and lower parts of the tube, although the flow again regains momentum as the spirals degenerate into turbulence. When $H = 10$ cm the flow between the rolls and the tube boundaries at $z = H/2$ is never completely blocked, and serves to distort the 'secondary rolls' and the resulting turbulence.

The motion eventually becomes three-dimensional. The onset and growth of transverse instability in the tube with $H = 3$ cm was recorded by taking photographs through a mirror placed above the tube which showed the tube 'in plan' whilst simultaneously, and with the same camera, photographing the tube directly from the side. The effects are best seen as a coloured layer of fluid of intermediate density became unstable. As remarked earlier, most of the dye is transferred into the rolls, and if the tube is viewed from above, a series of streaks of coloured fluid are seen lying across the tube (looking rather like billow clouds). The amalgamation of two rolls through their joining at one end and so producing a three-dimensional configuration, was found to occur when a wave crest in the initial growth of instability was not quite normal to the side walls of the tube. The part of the roll, formed by the roll-up of this wave, which was closest to an immediate neighbour usually interacted with this neighbour and began to wind round it, and usually the remainder of the roll eventually took part in this secondary roll-up. This sort of three-dimensional motion was not uncommon; some evidence of it was present in each experiment but there was no evidence of an associated periodic structure which might have resulted from a transverse wave motion being present in the apparatus. On one occasion a more complex interaction between three rolls was observed, one end of the central roll joining its left-hand neighbour and the other end joining the right. It was difficult to assess exactly when the motion could be regarded as three-dimensional and turbulent, but it appeared to be at some time after the complete formation of the rolls and

† The appearance of these was similar to the billows sometimes observed on the top of cumulus clouds in a strong wind (see, for example, Scorer & Wexler 1963, p. 42, figure 33).

was probably at a time of about $0{\cdot}45t_1$, and at a time after the spirals had begun to break up.

Careful measurement of the growth of the waves has been made and it is hoped that these, and other details, will be published later. The onset of instability occurs earlier if either α or $\Delta\rho$ is increased; the wavelength of the instability decreases as α or $\Delta\rho$ increase; the rate of growth of the waves increases as α and $\Delta\rho$ increase. If h_1 is the depth of the upper fluid (water) and h_2 the depth of the brine, it is found that as h_1/h_2 increases, the wavelength of the instability decreases and the time, t_1, increases. The waves are now no longer stationary on the flow but move in the direction of the shallower, faster moving fluid.

The persistence of waves growing at a particular part of the tube with $H = 10$ cm and the surprising repeatability of the development of instability at the location led us to suspect that there might be some irregularity in the tube at this place, and we found that the height H varied by a maximum of $0{\cdot}08$ cm over a distance of 5 or 6 cm there. The tube had been machined to have a side of 3 cm, but less care was taken in mounting the other sides and this resulted in the variations found. The effect of small disturbances was further demonstrated by the flow in the presence of the platinum wires used in measuring fluid speeds. A wave grew much earlier near the wires than those elsewhere, and the onset of instability in the remainder of the tube was delayed. Whether the wires initiate the instability or, through drag, cause a hydraulic jump, is not yet established.

4.2. *Two miscible fluids with a diffuse interface*

Some experiments have been made by tilting the tube after allowing time for diffusion between the brine and water. If τ is the time for diffusion, then a region of intermediate density and thickness of about $4(K\tau)^{\frac{1}{2}}$ results, where K is the molecular diffusivity of salt, about $1{\cdot}4\times10^{-5}\,\mathrm{cm\,sec^{-1}}$. The relative roles of viscosity and density variation in determining the velocity profile in the interfacial region may be measured by the ratio $(\nu t_1/K\tau)^{\frac{1}{2}}$, where t_1 is the time of onset of instability after the tube has been tilted. If this ratio is large then the velocity profile at the onset of instability is mainly determined by viscous diffusion; if it is small then the density distribution establishes the velocity distribution. It is found that t_1 increases as τ increases but that t_1/τ decreases. At small τ (4 min is the smallest time possible, as this is needed for the fluid to settle in the tube), the ratio is large (values of about 3 are typical). In the present apparatus the ratio is at least about unity, and is never small, and the effect of viscosity is therefore always important in determining the velocity profile. It is found that as τ increases, so the wavelength of the instability, λ, increases. In the tube of depth $H = 10$ cm the variation of λ with $\tau^{\frac{1}{2}}$ is almost linear so far as experiments have been continued, but the effect of the boundaries when $H = 3$ cm becomes apparent at wavelengths greater than $4{\cdot}5$ cm and the observed wavelengths are less than these found in the tube with $H = 10$ cm at the same τ. The minimum local Richardson number in the fluid at the onset of instability has been estimated in a number of cases. A typical value is $0{\cdot}09$. Instability has not been observed at values greater than $0{\cdot}25$.

4.3. *Immiscible fluids*

A few experiments have been made using paraffin (kerosene) and water. These indicate that the wave-number of the initial instability is greater than that predicted by the Kelvin–Helmholtz theory (k^*, see Chandrasekhar 1961, chapter XI (38)) and that the difference in velocity of the two liquids at the onset of instability is greater than the predicted critical value (Chandrasekhar 1961, chapter XI (40)). The flow is observed to become much more irregular than that observed when miscible fluids are used, and the spiral configuration has not been produced.

4.4. *Constant density gradient*

Several experiments have been made with a tilted tube containing a fluid of constant density gradient, the motions of the fluid being indicated by layers of dye introduced during filling. The parallel flow in the centre of the tube is always found to be interrupted eventually by the arrival of the surges from the tube ends; no instablity of the parallel flow itself has been observed although Richardson numbers just before the arrival of the surge are estimated to be as small as 0·014, and the Richardson number has been less than 0·25 for 6·8 sec in a total time of experiment of 9 sec.

5. Final remarks

A larger tube is being constructed and this will allow the continuation of experiments for a wider range of parameters, particularly those discussed in §§4.2 and 4.4 and a study of the non-accelerating flow in a horizontal tube produced by tilting the tube for a short time.

This investigation is part of a study of the breaking of internal waves (see also Thorpe 1968). Internal waves sometimes break as the result of self-induced shear which may locally reduce the Richardson number considerably. Such wave breaking has been observed in the Mediterranean off Malta (Woods 1968).

Appendix

We consider the same problem as that examined above in §2, but now we shall suppose that the fluid is viscous, but of constant coefficient of viscosity μ, and make the Boussinesq approximation. The equations of motion are now

$$\rho_0 \frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} - g\rho \sin\alpha + \mu \frac{\partial^2 u}{\partial z^2} \tag{A 1}$$

and

$$0 = -\frac{\partial p}{\partial z} - g\rho\cos\alpha, \tag{A 2}$$

where ρ_0 is the density at $z = 0$.

Reasoning as before we find that $\partial p/\partial x$ is a function of time, t, only and rearranging (A 1) and making the Laplace transformation we obtain

$$\frac{\partial^2 \bar{u}}{\partial z^2} - \frac{s\bar{u}}{\nu} = \frac{1}{\mu}\frac{\overline{\partial p}}{\partial x} + \frac{g\rho\sin\alpha}{\mu s}, \tag{A 3}$$

where
$$\bar{u}(z,s) = \int_0^\infty e^{-st} u(z,t)\,dt$$

is the Laplace transform of u, $\overline{\partial p/\partial x}$ is the Laplace transform of $\partial p/\partial x$ and $\nu = \mu/\rho_0$. This equation has the solution

$$\bar{u} = A\sinh rz + B\cosh rz - \frac{1}{s\rho_0}\frac{\overline{\partial p}}{\partial x} + \frac{g\sin\alpha}{\mu s}\int_0^z \rho(z_1)\sinh r(z-z_1)\,dz_1, \qquad \text{(A 4)}$$

where $r = (s/\nu)^{\frac{1}{2}}$. Three conditions are now applied to specify A, B and $\overline{\partial p/\partial x}$

$$u = 0 \quad \text{at} \quad z = \pm H/2 \quad \text{and} \quad \int_{-H/2}^{H/2} \bar{u}\,dz = 0,$$

which follow from the conditions of zero slip at the boundaries and zero flux across any plane $x = $ constant. With these conditions applied we find

$$\begin{aligned}\bar{u}(z,s) = \frac{g\sin\alpha}{2\mu s(rH\cosh(rH/2) - 2\sinh(rH/2))}\Big\{&[\chi(H/2) - \chi(-H/2)] \\ &\times [2 - Hr\coth(rH/2)]\sinh(rH/2) + [\chi(H/2)+\chi(-H/2)] \\ &\times [2\sinh rH/2 - rH\cosh rz] - 2\chi(z)[2\sinh(rH/2) - rH\cosh(rH/2)] \\ &+ 2r\int_{-H/2}^{H/2}\chi(z)\,dz[\cosh rz - \cosh(rH/2)]\Big\}, \qquad \text{(A 5)}\end{aligned}$$

where
$$\chi(z) = \int_0^z \rho(z_1)\sinh r(z-z_1)\,dz_1.$$

Application of the inverse Laplace transform and the initial condition that $u = 0$ at $t = 0$ for all z, then specifies u.

The solution for the flow in the limit as $t \to \infty$ may be found by solving the equations for steady flow

$$0 = -\frac{\partial p}{\partial x} - g\rho\sin\alpha + \frac{d}{dz}\left(\mu\frac{du}{dz}\right), \qquad \text{(A 6)}$$

$$0 = -\frac{\partial p}{\partial z} - g\rho\cos\alpha. \qquad \text{(A 7)}$$

(A 7) implies that $\partial p/\partial x$ is constant. Here we have taken both μ and ρ to vary with z. The solution is

$$u = C_0\int_0^z \frac{z_1}{\mu(z_1)}\,dz_1 + g\sin\alpha\int_0^z\left\{\frac{\int_0^{z_0}\rho(z_2)\,dz_2}{\mu(z_1)}\right\}dz_1 + C_1\int_0^z\frac{dz_1}{\mu} + C_2, \qquad \text{(A 8)}$$

where C_0, C_1 and C_2 are constants specified by the conditions $u = 0$ at $z = \pm H/2$ and $\int_{-H/2}^{H/2} u\,dz = 0$.

For example, if μ is constant and $\rho = \rho_0(1-\beta z)$, ρ_0, β constant, then

$$u = \frac{\beta g\rho_0\sin\alpha}{6\mu}z(H^2/4 - z^2). \qquad \text{(A 9)}$$

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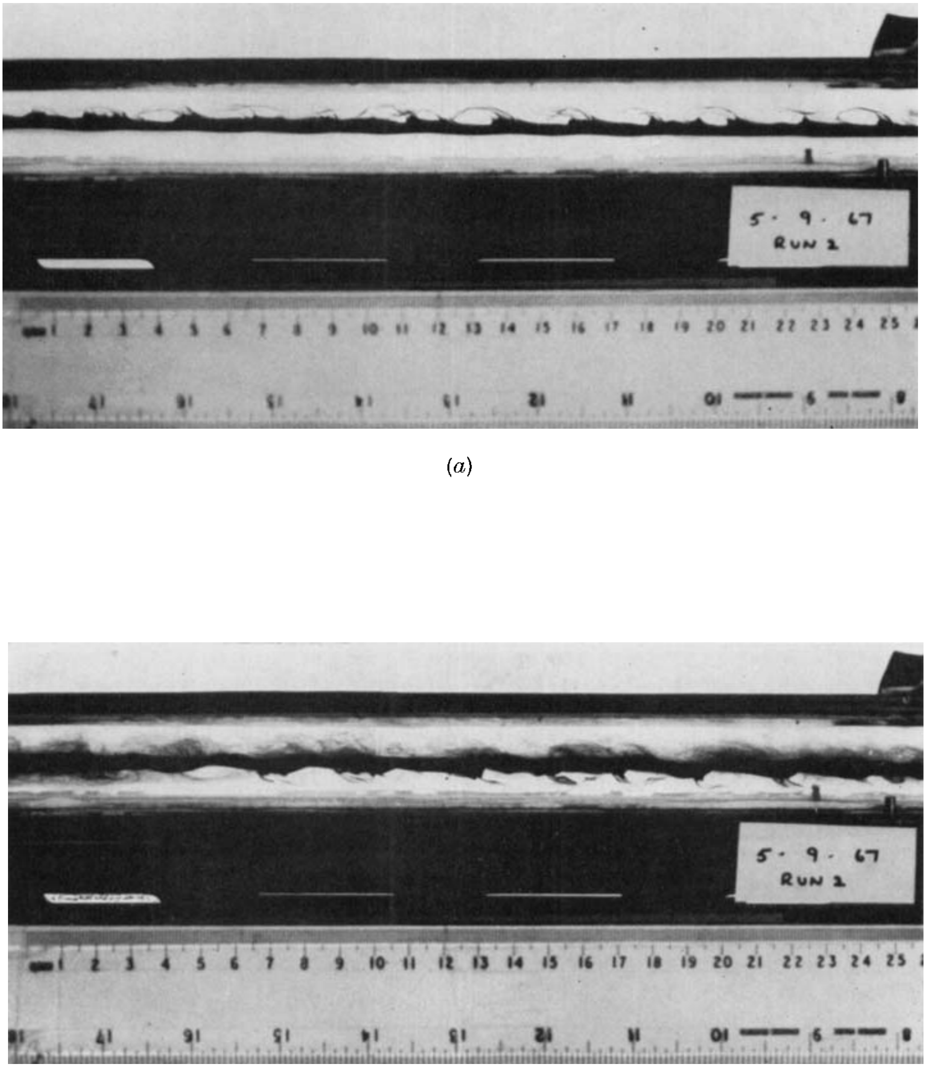

(*a*)

(*b*)

FIGURE 3. 'Cusped' breaking waves. The fluid in the upper part of the tube moves to the left and that in the bottom moves to the right. The dye belongs to fluid of intermediate density. The upper fluid is water and the lower brine. The density difference between these fluids is $9{\cdot}51 \times 10^{-2}$ gm c.c.$^{-1}$. (*a*) One train of waves moving to the left and mixing occurring in the upper liquid. (*b*) Two trains of waves moving through one another. The interface retains its identity in this sort of mixing process unlike that illustrated in figure 5. The scale is in centimetres.

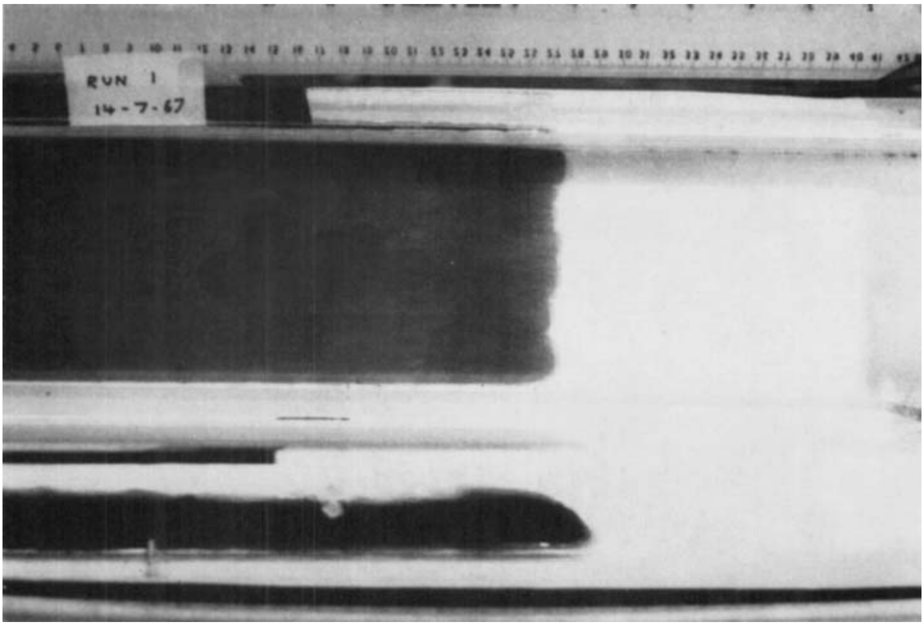

FIGURE 4. An advancing surge seen from the side (lower half of figure) and from above. The mirror reflexion of the scale is in inches and centimetres. The density difference between the fluids is $3{\cdot}34 \times 10^{-2}$ gm cm^{-3}.

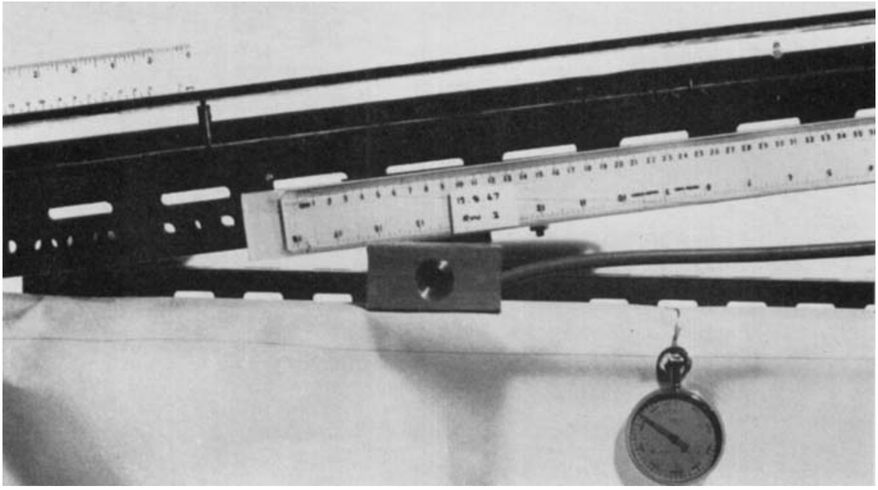

FIGURE 5 (*a*)

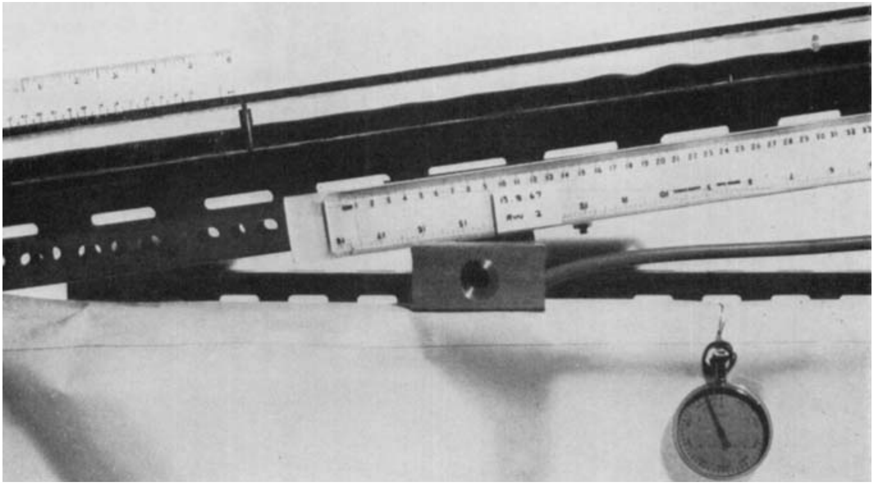

FIGURE 5 (*b*)

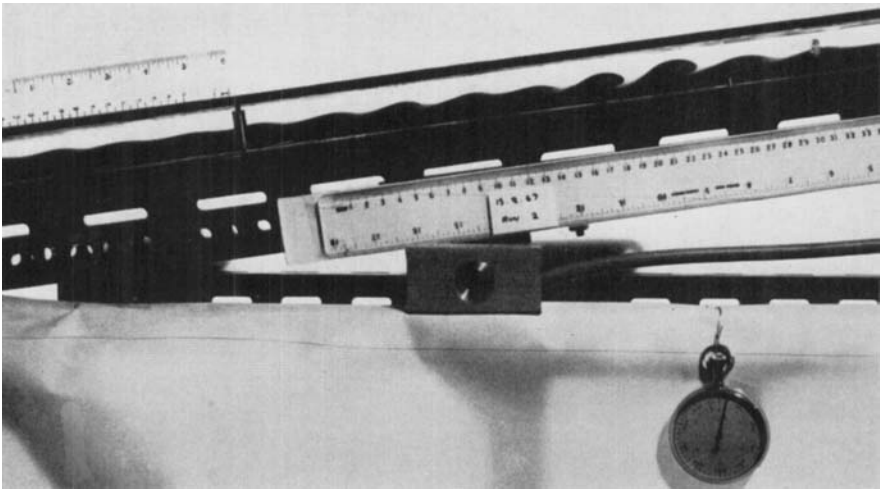

FIGURE 5 (c)

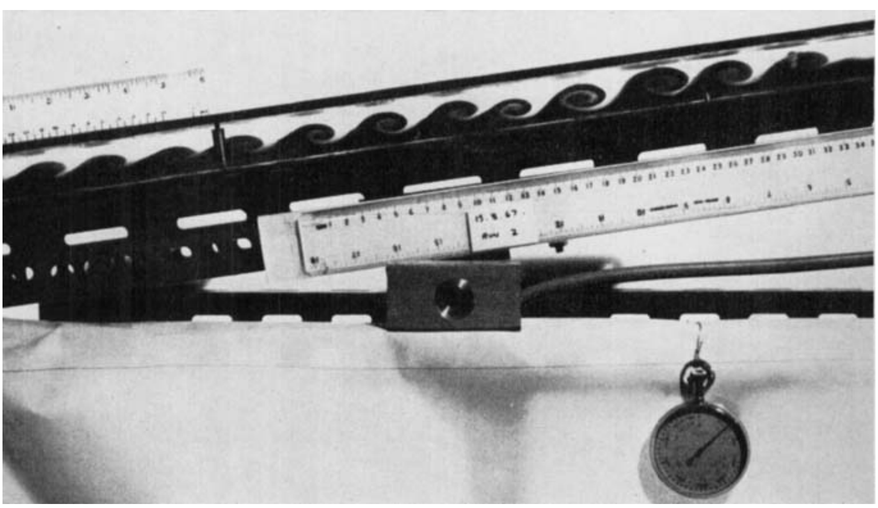

FIGURE 5 (d)

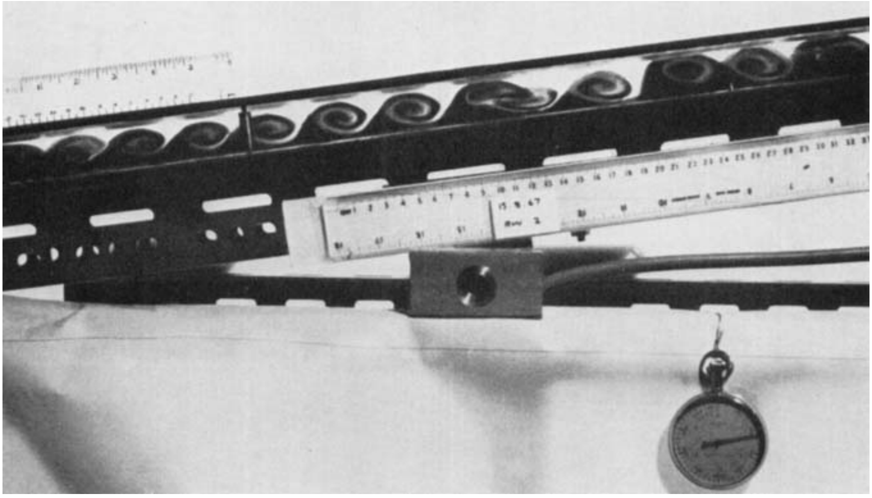

FIGURE 5 (*e*)

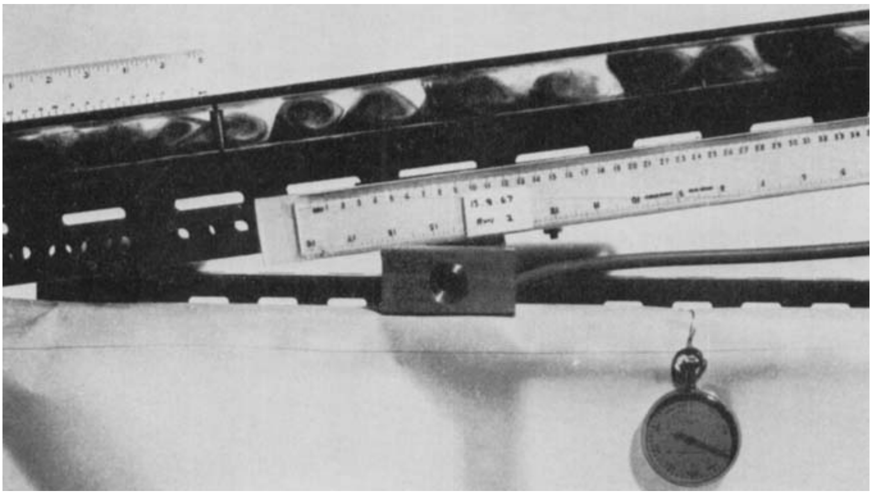

FIGURE 5 (*f*)

FIGURE 5, (*a*) to (*f*). A sequence of six photographs taken at approximately half-second intervals showing the growth of rolls at the interface between two fluids of equal depth in relative acceleration. The density difference between the fluids is $1{\cdot}56 \times 10^{-2}$ gm cm^{-3} and the tube height 3 cm. The scale is in centimetres and inches.